# Nonlinear Propagation of Incoherent Photons in a Radiation Background

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The nonlinear propagation of intense incoherent photons in a photon gas is considered. The photon-photon interactions are governed by a pair of equations comprising a wave-kinetic equation for the incoherent photons in the presence of the slowly varying energy density perturbations of sound-like waves, and an equation for the latter waves in a background where the photon coupling is caused by quantum electrodynamical effects. The coupled equations are used to derive a dispersion relation, which admits new classes of modulational instabilities of incoherent photons. The present instabilities can lead to fragmentation of broadband short photon pulses in astrophysical and laboratory settings.

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The advent of quantum electrodynamics (QED) is one of the major scientific achievements of the $20^{\text{th}}$ century. It is an experimentally well confirmed theory, and it has predicted a number of phenomena which were not expected in previous studies, e.g., the Casimir effect. Moreover, as opposed to the classical theory of Maxwell, electromagnetic waves can according to QED interact in the absence of a material mediator. Due to the possibility of exchanging virtual electron–positron pairs, there is the intriguing effect of photon–photon scattering, first discussed even before the interaction between light and matter was well understood [1, 2], and later derived within QED by Schwinger [3]. The fact that strong electromagnetic fields can interact in vacuum, opens up for very interesting applications in extreme astrophysical environments, such as magnetars [4], where photon splitting and lensing may take place [5, 6, 7, 8]. The prospect of direct detection of the effect has also been discussed in the literature, and a number of suggestions for experimental setups have been given, involving second harmonic generation [9], self-focusing [10], nonlinear wave mixing in cavities [11, 12] and waveguide propagation [13], respectively. Apart from the astrophysical regions where high fields exist, the rapid development of ultra-high laser fields [14, 15] and related laser-plasma techniques [16], gives hope that the critical field strengths will be produced in laboratories.

We shall below investigate the effect of incoherence of high-frequency photons propagating on a radiation fluid background. This will extend the work presented in Ref. [17], and investigated numerically in Ref. [18], to the random phase regime. For this purpose, a wave kinetic theory for high-frequency photons coupled to an acoustic wave equation for a radiation fluid is presented. It is shown that modulational instabilities are inherent in the system of equations. Moreover, in the limit of the slow time-variation approximation, we obtain a Vlasov equation with self-interaction for the high-frequency photons.

We consider an incoherent non-thermal high-frequency spectrum of photons. As will be shown, this spectrum will be able to interact with low-frequency acoustic-like perturbations. The high-frequency part is treated by means of a wave kinetic description, whereas the low-frequency part is described by an acoustic wave equation with a driver [17] which follows from a radiation fluid description. Let $N_k(t,\boldsymbol{r})$ denote the high frequency photon distribution function, normalised such that the corresponding number density is given by $n = \int N_k\, d^3k$. Then $N_k$ will satisfy the wave kinetic equation [19]

$$\frac{\partial N_k}{\partial t} + \boldsymbol{v}_g \cdot \frac{\partial N_k}{\partial \boldsymbol{r}} - \frac{\partial \omega_k}{\partial \boldsymbol{r}} \cdot \frac{\partial N_k}{\partial \boldsymbol{k}} = 0 \tag{1}$$

where $\boldsymbol{v}_g = \partial\omega_k/\partial\boldsymbol{k}$ is the group velocity, and

$$\omega_k = ck\left(1 - \tfrac{2}{3}\lambda\mathscr{E}\right), \tag{2}$$

where $\mathscr{E}$ is the radiation fluid density [17, 20], and $\lambda = 8\kappa$ or $14\kappa$, depending on the polarisation state of the photon. Here $\kappa \equiv 2\alpha^2\hbar^3/45m_e^4c^5 \approx 1.63\times10^{-30}\,\mathrm{ms^2/kg}$, $\alpha$ is the fine-structure constant, $2\pi\hbar$ the Planck constant, $m_e$ the electron mass, and $c$ the velocity of light in vacuum. The dispersion relation (2) is valid as long as there is no pair creation and the field strength is smaller than the QED critical field, i.e.,

$$\omega \ll m_ec^2/\hbar \quad \text{and} \quad |\boldsymbol{E}| \ll E_{\text{crit}} \equiv m_ec^2/e\lambda_c \tag{3}$$

respectively. Here $e$ is the elementary charge, $\lambda_c$ is the Compton wave length, and $E_{\text{crit}} \simeq 10^{18}\,\mathrm{V/m}$.

The high-frequency photons drive low-frequency acoustic perturbations according to [17]

$$\left(\frac{\partial^2}{\partial t^2} - \frac{c^2}{3}\nabla^2\right)\mathscr{E} = -\frac{2\lambda\mathscr{E}_0}{3}\left(\frac{\partial^2}{\partial t^2} + c^2\nabla^2\right)\int \hbar\omega_k N_k\, d^3k \tag{4}$$

where the constant $\mathscr{E}_0$ is the *background* radiation fluid energy density. This hybrid description, where the high-frequency part is treated kinetically, and the low-frequency part is described within a fluid theory, applies when the mean-free path between photon-photon

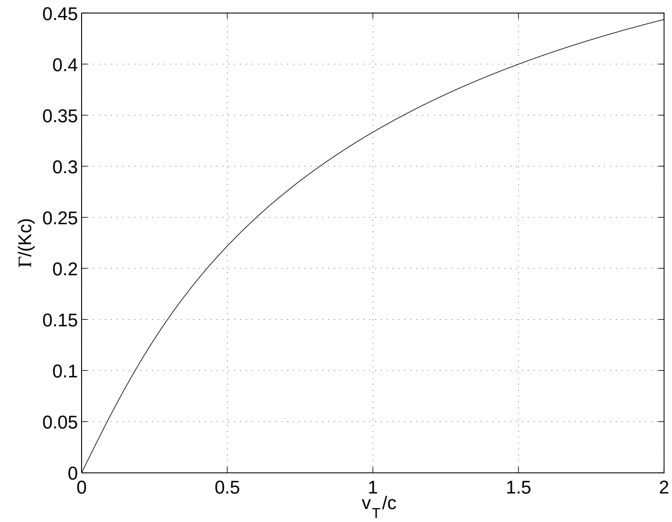


FIG. 1: The transverse instability for the mono-energetic case. $\varGamma/Kc$ plotted as a function of $v_T/c$, as given in Eq. (8).

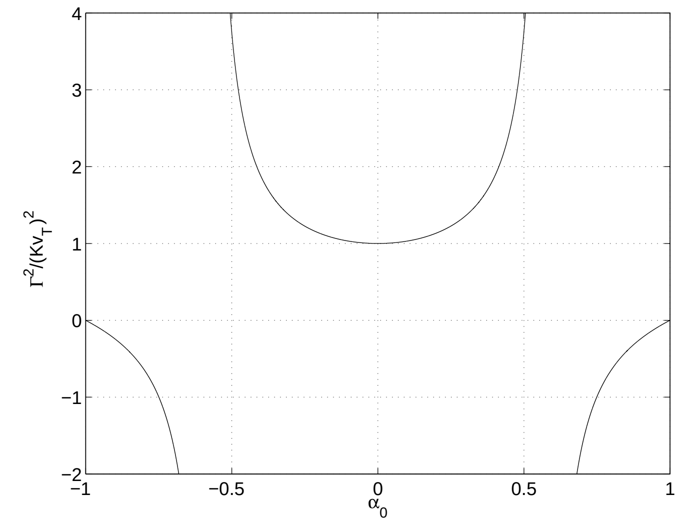


FIG. 2: $(\varGamma/Kv_T)^2$, according to Eq. (9), plotted as a function of $\alpha_0 = \cos\theta_0$ in the mono-energetic case.

collisions is shorter than the wavelengths of the low-frequency perturbations. We note that the intensity $I_k = \hbar\omega_k N_k/\epsilon_0$ satisfies Eq. (1), and is normalised such that $\langle |E|^2\rangle = \int I_k\, d^3k$, where $E$ is the high- frequency electric field strength, and $\epsilon_0$ is the dielectric constant of vacuum. The equations presented here resemble the photon–electron system in Ref. [21], where the interaction between random phase photons and sound waves in an electron–positron plasma has been investigated.

Next we consider a small low-frequency long wavelength perturbation of a homogeneous background spectrum, i.e. $N_k = N_{k0} + N_{k1}\exp[i(Kz - \varOmega t)]$, $N_{k1} \ll N_{k0}$, and $\mathscr{E} = \mathscr{E}_1 \exp[i(Kz - \varOmega t)]$ and linearise our equations. We thus obtain (using expression (2) for $\omega_k$, and introducing $\hat{\boldsymbol{k}} = \boldsymbol{k}/k$)

$$N_{k1} = \frac{2\lambda k \mathscr{E}_1}{3}\frac{Kc}{\varOmega - Kc\hat{\boldsymbol{k}}\cdot\hat{\boldsymbol{z}}}\hat{\boldsymbol{z}}\cdot\frac{\partial N_{k0}}{\partial \boldsymbol{k}} \tag{5a}$$

and

$$\mathscr{E}_1 = -\frac{2\lambda c\hbar\mathscr{E}_0}{3}\frac{\varOmega^2 + K^2c^2}{\varOmega^2 - K^2c^2/3}\int kN_{k1}\, d^3k, \tag{5b}$$

which, when combined, give the nonlinear dispersion relation

$$1 = -\frac{\mu K}{3}\frac{\varOmega^2 + K^2c^2}{\varOmega^2 - K^2c^2/3}\int \frac{k^2}{\varOmega - Kc\hat{\boldsymbol{k}}\cdot\hat{\boldsymbol{z}}}\hat{\boldsymbol{z}}\cdot\frac{\partial N_{k0}}{\partial \boldsymbol{k}}\, d^3k, \tag{6}$$

where we have introduced the constant $\mu = \frac{4}{3}\lambda^2c^2\hbar\mathscr{E}_0$. As was found in Ref. [21, 22], we may have growth for a large class of background distributions $N_{k0}$.

(a) For a mono-energetic high frequency background, we have $N_{k0} = n_0\delta(\boldsymbol{k} - \boldsymbol{k}_0)$. The nonlinear dispersion relation (6) then reduces to

$$(\varOmega^2 - K^2c^2/3)(\varOmega - Kc\cos\theta_0)^2 = \tfrac{1}{3}\mu n_0 k_0 K \times \times(\varOmega^2 + K^2c^2)[Kc + (2\varOmega - 3Kc\cos\theta_0)\cos\theta_0], \tag{7}$$

where we have introduced $\cos\theta_0 \equiv \hat{\boldsymbol{k}_0}\cdot\hat{\boldsymbol{z}}$. This mono-energetic background has a transverse instability when $\theta_0 = \pi/2$, with the growth rate

$$\varGamma = \frac{1}{\sqrt{6}}Kc\left[\sqrt{\left(\frac{v_T}{c}\right)^4 + 14\left(\frac{v_T}{c}\right)^2 + 1} - \left(\frac{v_T}{c}\right)^2 - 1\right]^{1/2}, \tag{8}$$

where $\varGamma \equiv -i\varOmega$, and $v_T^2 \equiv \mu n_0 k_0 c$, and $v_T$ is a characteristic speed of the system. The expression in the square bracket is positive definite. In Fig. 1 we have plotted $\varGamma/Kc$ as a function of $v_T/c$.

In fact, under most circumstances, $v_T \ll c$. Using the expression (7), we then have two branches. The branch corresponding to $\varOmega \simeq Kc/\sqrt{3}$ is always stable for small $v_T$, while for the branch corresponding to $\varOmega \simeq Kc\cos\theta_0$ we obtain the growth rate

$$\varGamma = Kv_T\sqrt{\frac{1-\cos\theta_0}{1-3\cos\theta_0}}, \tag{9}$$

which is consistent with (8) in the limit $\theta_0 \to \pi/2$. In Fig. 2, the behaviour of the growth rate (9) is depicted.

(b) The high-frequency photons have generally a spread in momentum space. For simplicity we here choose the background intensity distribution as a shifted Gaussian, i.e.

$$I_{k0} = \frac{\mathscr{I}_0}{\pi^{3/2}k_W^3}\exp\left[-\frac{(\boldsymbol{k} - \boldsymbol{k}_0)^2}{k_W^2}\right], \tag{10}$$

where $\mathscr{I}_0 = \langle |E_0|^2 \rangle$ is the (constant) background intensity and $k_W$ is the width of the distribution around $\boldsymbol{k}_0$. Then the dispersion relation is

$$1 = -\frac{b^2}{k_W^5}\frac{\eta^2+1}{\eta^2-1/3}\int\left[\frac{k(k_0\cos\theta_0 - k\cos\theta)}{\eta-\cos\theta}\right. \\ \left.\times\exp\left(-\frac{k^2-2\boldsymbol{k}\cdot\boldsymbol{k}_0}{k_W^2}\right)\right]d^3k, \quad (11)$$

where $b^2 = (4/9\pi^{3/2})\lambda^2\epsilon_0\mathscr{E}_0\mathscr{I}_0\exp(-k_0^2/k_W^2)$ and $\eta \equiv \Omega/Kc$.

Assuming that the deviation of $\boldsymbol{k}_0$ from the $\hat{\boldsymbol{z}}$-axis is small, and that $\delta \equiv k_0/k_W \ll 1$, we can integrate Eq. (11), keeping terms linear in $\delta$, to obtain

$$1 \simeq -\pi b^2\frac{\eta^2+1}{\eta^2-1/3}\left[\frac{3\sqrt{\pi}}{2} + 8\delta\eta\cos\theta_0\right. \\ \left. + \left(\delta\cos\theta_0 - \frac{3\sqrt{\pi}}{4}\eta - 4\delta\eta\cos\theta_0\right)(2\,\mathrm{arctanh}\,\eta - i\pi)\right], \quad (12)$$

for $0 < \eta < 1$. Thus, we see that the non-zero width of the distribution complicates the characteristic behaviour of the dispersion relation by a considerable amount. It is clear though, that the width will introduce a reduction of the growth rate, as compared to the mono-energetic case.

We may also look at the case when the time-dependence is weak, i.e. $\partial^2\mathscr{E}/\partial t^2 \ll c^2\nabla^2\mathscr{E}$, such that Eq. (4) yields

$$\mathscr{E} = 2\lambda\mathscr{E}_0\int\hbar\omega_k N_k\,d^3k. \quad (13)$$

Upon using Eq. (13) in (2), we find

$$\frac{\partial\omega_k}{\partial\boldsymbol{r}} = -\mu k\frac{\partial}{\partial\boldsymbol{r}}\int k' N_{k'}\,d^3k' \quad (14)$$

Hence Eq. (1) becomes

$$\frac{\partial N_k}{\partial t} + \boldsymbol{v}_g\cdot\frac{\partial N_k}{\partial\boldsymbol{r}} + \mu k\left(\frac{\partial}{\partial\boldsymbol{r}}\int k' N_{k'}\,d^3k'\right)\cdot\frac{\partial N_k}{\partial\boldsymbol{k}} = 0, \quad (15)$$

which in the one-dimensional case reduces to

$$\frac{\partial N_k}{\partial t} + v_g\frac{\partial N_k}{\partial x} + \mu k\left(\frac{\partial}{\partial x}\int k' N_{k'}\,dk'\right)\frac{\partial N_k}{\partial k} = 0. \quad (16)$$

A similar equation may of course be derived for the intensity $I_k$.

Equation (16) gives the evolution of high-frequency photons on a slowly varying background radiation fluid, and it may be used to analyse the long term behaviour of amplitude modulated intense short incoherent laser pulses.

We have considered the nonlinear propagation of randomly distributed intense short photon pulses in a photon gas. The photon-photon interactions are described by means of a QED model, in which an ensemble of incoherent photon pulses is governed by a wave kinetic equation where the coupling between the intense photon pulses and the sound-like waves of the photon gas is due to slow variations of the sound wave energy distribution. The intense photon pressure, in turn, modifies the sound wave propagation. The wave kinetic and the sound wave equations form a closed system, which has been used to derive a dispersion relation. By choosing appropriate spectra for short pulse photons, we analyze the dispersion relation to show the existence of new classes of modulational instabilities. The latter can cause fragmentation of incoherent photon pulses in astrophysical contexts and in forthcoming experiments using very intense short laser pulses.


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